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Mechanical behavior of functionally graded material plates under transverse load—Part I: Analysis

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Abstract

An elastic, rectangular, and simply supported, functionally graded material (FGM) plate of medium thickness subjected to transverse loading has been investigated. The Poisson's ratios of the FGM plates are assumed to be constant, but their Young's moduli vary continuously throughout the thickness direction according to the volume fraction of constituents defined by power-law, sigmoid, or exponential function. Based on the classical plate theory and Fourier series expansion, the series solutions of power-law FGM (simply called P-FGM), sigmoid FGM (S-FGM), and exponential FGM (E-FGM) plates are obtained. The analytical solutions of P-, S- and E-FGM plates are proved by the numerical results of finite element method. The closed-form solutions illustrated by Fourier series expression are given in Part I of this paper. The closed-form and finite element solutions are compared and discussed in Part II of this paper. Results reveal that the formulations of the solutions of FGM plates and homogeneous plates are similar, except the bending stiffness of plates. The bending stiffness of a homogeneous plate is $Eh^3/12(1-\nu^2)$, while the expressions of the bending stiffness of FGM plates are more complicated combination of material properties.

Keywords: FGM plate; Fourier series expansion; Analytical solution

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1. Introduction

Traditional composites comprised of two different materials have been widely used to satisfy the high performance demands. However, stress singularities in such composites may occur at the interface between two different materials, due to the mismatch of materials. Especially, in a high-temperature environment, for example in the engine combustion chamber of an air vehicle or a nuclear fusion reaction container, the relatively higher mismatch in thermal expansion coefficients will induce high residual stresses. Consequently, the composite may incur cracking or debonding. Therefore, the concept of functionally graded material (FGM) was introduced to satisfy the demand of ultra-high-temperature environment and to eliminate the stress singularities (Niino and Maeda, 1990; Hirano and Yamada, 1988).

An FGM can be prepared by continuously changing the constituents of multi-phase materials in a predetermined volume fraction of the constituent material (Khor et al., 1997; Kwon and Crimp, 1997; Nogata, 1997). Due to the continuous change in material properties of an FGM, the interfaces between two materials disappear but the characteristics of two or more different materials of the composite are preserved. Subsequently the stress singularity at the interface of a composite can be eliminated and thus the bonding strength is enhanced. Studies reveal that the thermal residual stresses can be significantly relaxed by using an FGM (Lee and Erdogan, 1995; Drake et al., 1993; Chung and Chi, 2001).

Power-law function (Jin and Paulino, 2001; Yung and Munz, 1996), and exponential function (Jin and Batra, 1996; Delale and Erdogan, 1983; Gu and Asaro, 1997; Erdogan and Wu, 1996; Jin and Noda, 1994; Erdogan and Chen, 1998) are commonly used to describe the variations of material properties of FGMs. However, in both power-law and exponential functions, the stress concentrations appear in one of the interfaces in which the material is continuous but rapidly changing. Therefore, Chung and Chi (2001) proposed a sigmoid FGM, which is composed of two power-law functions to define a new volume fraction. Chi and Chung (2002) indicated that the use of a sigmoid FGM can significantly reduce the stress intensity factors of a cracked body.

Because of the wide material variations and applications of FGMs, literatures corresponding to FGMs in the material constituent (Chi and Chung, 2003; Bao and Wang, 1995; Suresh and Mortensen, 1998), fracture mechanics (Jin and Batra, 1996; Delale and Erdogan, 1983; Gu and Asaro, 1997; Cai and Bao, 1998; Jin and Paulino, 2001; Erdogan and Wu, 1996), and processing (Kwon and Crimp, 1997; Kesler et al., 1997) have been rapidly increased in the last 10 years. Many researchers are devoted to understand the mechanics and mechanism of FGMs to offer an optimum profile for designers. The FGM may be applied to plate structure as a thermal barrier. The metal-ceramic composite plates are widely used in aircrafts, space vehicles, reactor vessels, and other engineering applications. If a high external pressure is applied to the composite plate structure, the high stresses occurred in the structure will affect its integrity and the structure is susceptible to failure. Therefore, understanding the mechanical behavior of an FGM plate is very important to asses the safety of the plate structure. Woo and Meguid (2001) applied the Karman theory for large deformation to obtain the analytical solution for the plates and shell under transverse mechanical loads and a temperature field. Praveen and Reddy (1998) investigated the static and dynamic responses of functionally graded ceramic–metal plates by using a plate finite element that accounts for the transverse shear strains, rotary inertia and moderately large rotations in the Von Karman sense. He et al. (2001) studied the vibration control of the FGM plates with integrated piezoelectric sensors and actuators by a finite element formulation based on the classical laminated plate theory. Elastic bifurcation buckling of FGM plates under in-plane compressive loading was studied by Feldman and Aboudi (1997), based on a combination of micromechanical and structural approaches.

In this study, a simply supported elastic rectangular FGM plate subjected to transverse loadings is considered. The material properties of the FGM plates are assumed to change continuously throughout the thickness of the plate, according to the volume fraction of the constituent materials based on the power-law, exponential, or sigmoid functions. In part I of this paper, the series solutions of the FGM plates are obtained by expanding the transverse load into Fourier series expansion, and then the sinusoidal terms of the Fourier series are solved. In part II, the analytical solutions of the FGM plates are proved by the numerical results of finite element method using MARC software.

2. Material gradient of FGM plates

The functionally graded material (FGM) can be produced by continuously varying the constituents of multi-phase materials in a predetermined profile. The most distinct features of an FGM are the non-uniform microstructures with continuously graded macroproperties. An FGM can be defined by the variation in the volume fractions. Most researchers use the power-law function, exponential function, or sigmoid function to describe the volume fractions. Therefore, FGM plates with power-law, exponential, or sigmoid function will be considered in this paper.

Consider an elastic rectangular plate. As shown in Fig. 1, coordinates x and y define the plane of the plate, whereas the z-axis originated at the middle surface of the plate is in the thickness direction. The material properties, Young's modulus and the Poisson's ratio, on the upper and lower surfaces are different but are preassigned according to the performance demands. However, the Young's modulus and Poisson's ratio of the plates vary continuously only in the thickness direction (z-axis) i.e., $E = E(z)$, $v = v(z)$. It is called functionally graded material (FGM) plates. Delale and Erdogan (1983) indicated that the effect of Poisson's ratio on the deformation is much less than that of Young's modulus. Thus, Poisson's ratio of the plates is assumed to be constant. However, the Young's moduli in the thickness direction of the FGM plates vary with power-law functions (P-FGM), exponential functions (E-FGM), or with sigmoid functions (S-FGM).

2.1. The material properties of P-FGM plates

The volume fraction of the P-FGM is assumed to obey a power-law function:

$$g(z) = \left(\frac{z + h/2}{h}\right)^p \tag{1}$$

where p is the material parameter and h is the thickness of the plate. Once the local volume fraction $g(z)$ has been defined, the material properties of a P-FGM can be determined by the rule of mixture (Bao and Wang, 1995):

$$E(z) = g(z)E_1 + [1 - g(z)]E_2 \tag{2}$$

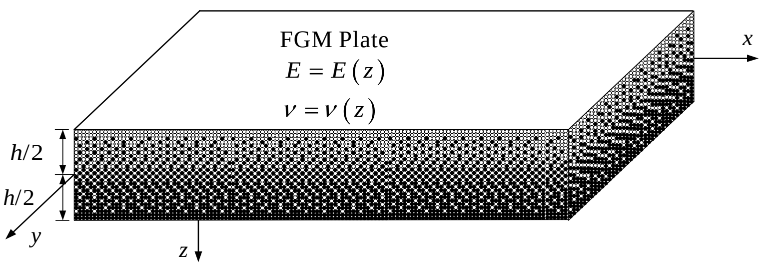

Fig. 1. The geometry of an FGM plate.

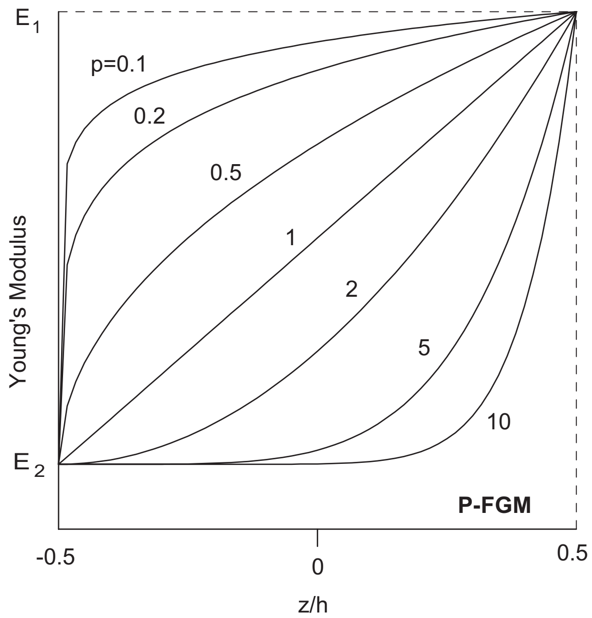

Fig. 2. The variation of Young's modulus in a P-FGM plate.

where E_1 and E_2 are the Young's moduli of the lowest ($z = h/2$) and top surfaces ($z = -h/2$) of the FGM plate, respectively. The variation of Young's modulus in the thickness direction of the P-FGM plate is depicted in Fig. 2, which shows that the Young's modulus changes rapidly near the lowest surface for $p > 1$, and increases quickly near the top surface for $p < 1$.

2.2. The material properties of S-FGM plates

In the case of adding an FGM of a single power-law function to the multi-layered composite, stress concentrations appear on one of the interfaces where the material is continuous but changes rapidly (Lee and Erdogan, 1995; Bao and Wang, 1995). Therefore, Chung and Chi (2001) defined the volume fraction using two power-law functions to ensure smooth distribution of stresses among all the interfaces. The two power-law functions are defined by:

$$g_1(z) = 1 - \frac{1}{2}\left(\frac{h/2 - z}{h/2}\right)^p \quad \text{for } 0 \leqslant z \leqslant h/2 \tag{3a}$$

$$g_2(z) = \frac{1}{2}\left(\frac{h/2 + z}{h/2}\right)^p \quad \text{for } -h/2 \leqslant z \leqslant 0 \tag{3b}$$

By using the rule of mixture, the Young's modulus of the S-FGM can be calculated by:

$$E(z) = g_1(z)E_1 + [1 - g_1(z)]E_2 \quad \text{for } 0 \leqslant z \leqslant h/2 \tag{4a}$$

$$E(z) = g_2(z)E_1 + [1 - g_2(z)]E_2 \quad \text{for } -h/2 \leqslant z \leqslant 0 \tag{4b}$$

Fig. 3 shows that the variation of Young's modulus in Eqs. (4a) and (4b) represents sigmoid distributions, and this FGM plate is thus called a sigmoid FGM plate (S-FGM plates).

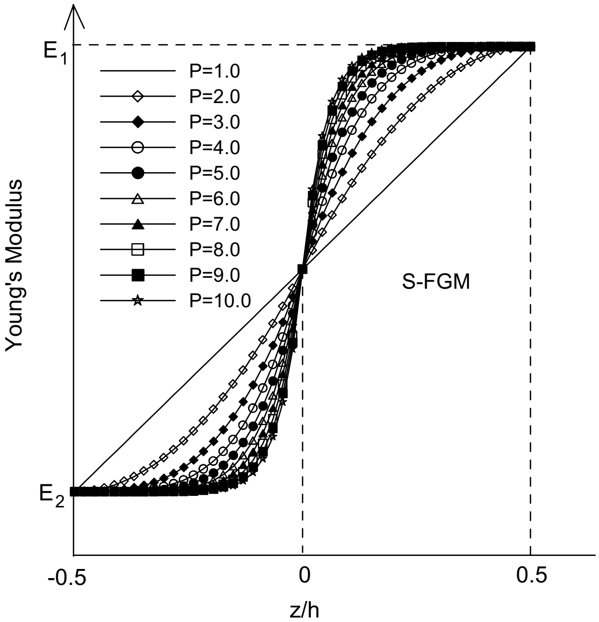

Fig. 3. The variation of Young's modulus in an S-FGM plate.

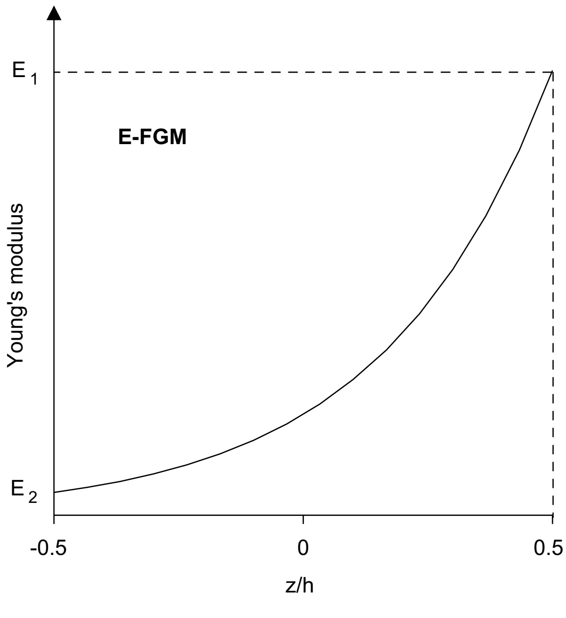

Fig. 4. The variation of Young's modulus in an E-FGM plate.

2.3. The material properties of E-FGM plates

Many researchers used the exponential function to describe the material properties of FGMs as follows (Delale and Erdogan, 1983)

$$E(z) = A\,e^{B(z+h/2)} \tag{5a}$$

with

$$A = E_2 \quad \text{and} \quad B = \frac{1}{h}\ln\left(\frac{E_1}{E_2}\right) \tag{5b}$$

The material distribution in the thickness direction of the E-FGM plates is plotted in Fig. 4.

3. Governing equations of rectangular FGM plates

A linearly-elastic, medium-thick, rectangular FGM plate subjected to transverse load is considered. It is assumed that medium-thick FGM plate has a uniform thickness h and the thickness h is in the range $1/20 \sim 1/100$ of its span approximately. The deformations and the stresses of the FGM plate are based upon the following assumptions:

1. Line elements perpendicular to the middle surface of the plate before deformation remain normal and unstretched after deformation.
2. The deflections of the FGM plate is small in comparison with its thickness h, such that the linear strain-displacement relations are valid.
3. The normal stress in the thickness direction can be neglected because the thickness which is assumed in the range $1/20 \sim 1/100$ of its span is small.
4. For the non-homogeneous elastic FGM plate, the Young's modulus and Poisson's ratio of the FGM plate are functions of the spatial coordinate z.

3.1. The stress field of FGM plate

According to the assumption 1, a point A in the FGM plate with a distance z to the middle surface will move to point A' after deformation (see Fig. 5). Consequently the transverse strain components ε_{zz}, γ_{xz}, and γ_{yz} are negligibly small. Therefore, the displacements at the point A in the x, y and z directions are

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w}{\partial x} \tag{6a}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w}{\partial y} \tag{6b}$$

$$w(x,y,z) = w_0(x,y) \tag{6c}$$

where $u_0(x,y)$, $v_0(x,y)$, $w_0(x,y)$ are the displacements at the middle surface. Under the assumption of small deformation, the strain field of the FGM plate is

$$\varepsilon_x = \frac{\partial u}{\partial x} = \varepsilon_{x_0} - z\frac{\partial^2 w}{\partial x^2} \tag{7a}$$

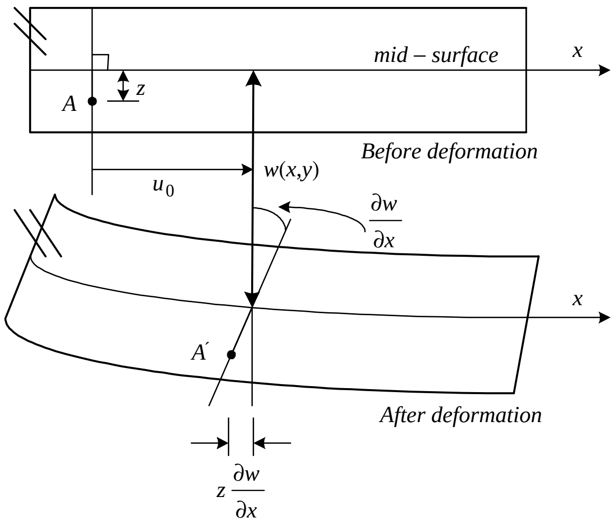

Fig. 5. The deformed configuration of an FGM plate.

$$\varepsilon_y = \frac{\partial v}{\partial y} = \varepsilon_{y_0} - z\frac{\partial^2 w}{\partial y^2} \tag{7b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy_0} - 2z\frac{\partial^2 w}{\partial x \partial y} \tag{7c}$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \tag{7d}$$

It is known that the neglect of the transverse shear deformations (see Eq. (7d)) can lead to significant errors when applied to moderately thick plate with thickness larger than 0.1 of span (Tauchert, 1986). However, Shames and Dym (1985) indicated that for a plate with the thickness less than 0.1 of its span, the classical theory of plates is expected to give good results. In this paper, the thickness of the medium-thick FGM plate is assumed to be in the range $1/20 \sim 1/100$ of its span, therefore the transverse shear deformations are negligible.

Based on the assumptions (3) and (4) the stress–strain relation for plane stress for an FGM plate is

$$\sigma_x = \frac{E(z)}{1 - v(z)^2}\left\{\varepsilon_{x_0} + v(z)\varepsilon_{y_0} - z\left[\frac{\partial^2 w}{\partial x^2} + v(z)\frac{\partial^2 w}{\partial y^2}\right]\right\} \tag{8a}$$

$$\sigma_y = \frac{E(z)}{1 - v(z)^2}\left\{\varepsilon_{y_0} + v(z)\varepsilon_{x_0} - z\left[\frac{\partial^2 w}{\partial y^2} + v(z)\frac{\partial^2 w}{\partial x^2}\right]\right\} \tag{8b}$$

$$\tau_{xy} = \frac{E(z)}{1 - v(z)^2}\left(\frac{1 - v(z)}{2}\right)\left[\gamma_{xy_0} - 2z\frac{\partial^2 w}{\partial x \partial y}\right] \tag{8c}$$

3.2. The axial forces, shear forces, and the bending moment of FGM plate

The stress resultants per unit length of the middle surface are defined by integrating stresses along the thickness. The inplane axial forces N_x, N_y, and N_{xy} are defined as

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, dz, \quad N_y = \int_{-h/2}^{h/2} \sigma_y \, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, dz \tag{9}$$

The transverse shear forces V_x and V_y are defined as

$$V_x = \int_{-h/2}^{h/2} \tau_{xz} \, dz, \quad V_y = \int_{-h/2}^{h/2} \tau_{yz} \, dz \tag{10}$$

and the bending moments M_x, M_y and M_{xy} are defined as

$$M_x = \int_{-h/2}^{h/2} z\sigma_x \, dz, \quad M_y = \int_{-h/2}^{h/2} z\sigma_y \, dz, \quad M_{xy} = \int_{-h/2}^{h/2} z\tau_{xy} \, dz \tag{11}$$

The sign conventions of the axial and shear forces are illustrated in Fig. 6(a) and that of the bending moment are shown in Fig. 6(b).

By substituting of Eq. (8) to Eqs. (9) and (11), we obtain the axial forces and the bending moments in the matrix forms as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x_0} \\ \varepsilon_{y_0} \\ \gamma_{xy_0} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{12}$$

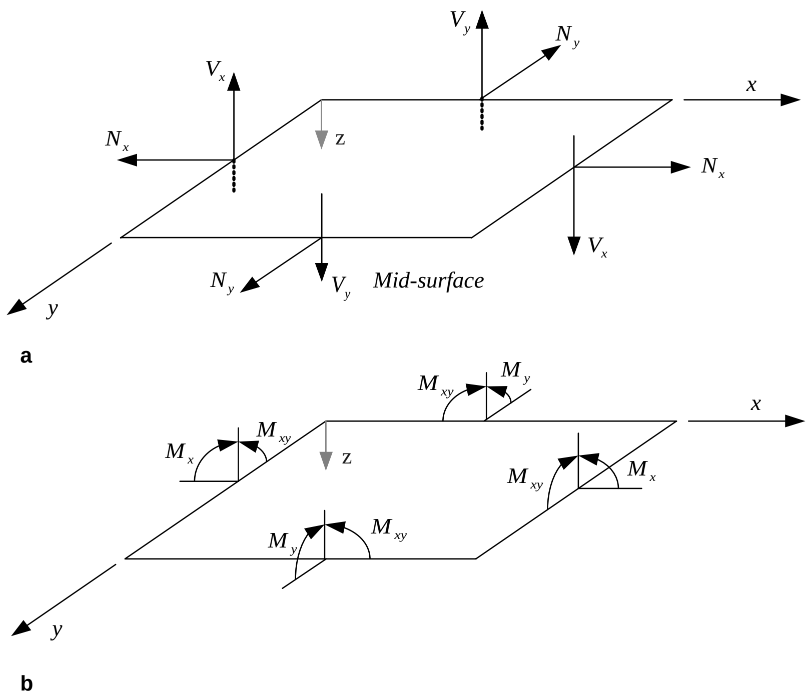

Fig. 6. (a) The positive directions of the axial and shear forces in FGM plates and (b) moments in FGM plates.

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x_0} \\ \varepsilon_{y_0} \\ \gamma_{xy_0} \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{13}$$

where the coefficients of A_{ij}, B_{ij} and C_{ij} are the integration of the material properties of the FGM plate and they are

$$A_{11} = \int_{-h/2}^{h/2} \frac{E(z)}{1-\nu(z)^2}\,\mathrm{d}z, \quad A_{12} = \int_{-h/2}^{h/2} \frac{E(z)\nu(z)}{1-\nu(z)^2}\,\mathrm{d}z, \quad B_{11} = \int_{-h/2}^{h/2} \frac{zE(z)}{1-\nu(z)^2}\,\mathrm{d}z, \quad B_{12} = \int_{-h/2}^{h/2} \frac{zE(z)\nu(z)}{1-\nu(z)^2}\,\mathrm{d}z,$$
$$C_{11} = \int_{-h/2}^{h/2} \frac{z^2E(z)}{1-\nu(z)^2}\,\mathrm{d}z, \quad C_{12} = \int_{-h/2}^{h/2} \frac{z^2E(z)\nu(z)}{1-\nu(z)^2}\,\mathrm{d}z, \quad A_{66} = \int_{-h/2}^{h/2} \frac{E(z)}{1-\nu(z)^2}\left(\frac{1-\nu(z)}{2}\right)\mathrm{d}z,$$
$$B_{66} = \int_{-h/2}^{h/2} \frac{zE(z)}{1-\nu(z)^2}\left(\frac{1-\nu(z)}{2}\right)\mathrm{d}z, \quad C_{66} = \int_{-h/2}^{h/2} \frac{z^2E(z)}{1-\nu(z)^2}\left(\frac{1-\nu(z)}{2}\right)\mathrm{d}z \tag{14}$$

3.3. The equilibrium and compatibility equations of FGM plate

Assume that FGM plate is subjected to the distributed loads q_x, q_y and q_z along the x, y and z directions. Consider a small solid element with dimensions dx, dy and dz. All forces acting on the small element are shown in Fig. 7. When the element is in equilibrium, the resultant forces in the x-direction must be zero, i.e.

$$\left(N_x + \frac{\partial N_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y + \left(N_{yx} + \frac{\partial N_{yx}}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x - N_x\,\mathrm{d}y - N_{yx}\,\mathrm{d}x + q_x\,\mathrm{d}x\,\mathrm{d}y = 0$$

or

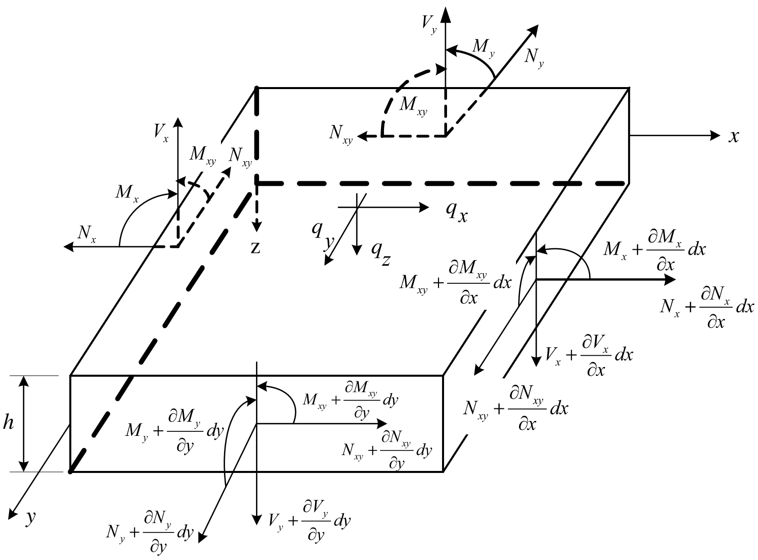

Fig. 7. The forces in a small element dxdydz of an FGM plate.

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}+q_x=0 \tag{15}$$

Similarly, the zero forces in y- and z-directions yield:

$$\frac{\partial N_{yx}}{\partial x}+\frac{\partial N_y}{\partial y}+q_y=0 \tag{16}$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+q_z=0 \tag{17}$$

Moreover, the zero resultant moments in the y- and x-directions give two equilibrium equations:

$$V_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y} \tag{18}$$

$$V_y=\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y} \tag{19}$$

The substituting of Eqs. (18) and (19) to Eq. (17) gives the equilibrium equation of FGM plate in terms of bending moments:

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=-q_z(x,y) \tag{20}$$

There are three moment components in Eq. (20), therefore the consideration of deformation is required. By applying Eqs. (13) and (20) can be rewritten as

$$B_{11}\left(\frac{\partial^2\varepsilon_{x_0}}{\partial x^2}+\frac{\partial^2\varepsilon_{y_0}}{\partial y^2}\right)+B_{12}\left(\frac{\partial^2\varepsilon_{x_0}}{\partial y^2}+\frac{\partial^2\varepsilon_{y_0}}{\partial x^2}\right)+2B_{66}\frac{\partial^2\gamma_{xy_0}}{\partial x\partial y}-C_{11}\frac{\partial^4 w}{\partial x^4}-(2C_{12}+4C_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}$$
$$-C_{11}\frac{\partial^4 w}{\partial y^4}=-q_z(x,y) \tag{21}$$

It can be seen from Eq. (21) that the strains at middle surface, ε_{x_0}, ε_{y_0} and γ_{xy_0}, and the deflection w are coupled. This phenomenon is different from the homogeneous plate in which the strains of middle surface and the deflection are uncoupled.

If the FGM plate is only subjected to the transverse load q_z, i.e., $q_x=0$, $q_y=0$, then the inplane equations (15) and (16) can be solved in terms of a stress function $\phi(x,y)$ which is defined by

$$N_x=\frac{\partial^2\phi}{\partial y^2},\quad N_y=\frac{\partial^2\phi}{\partial x^2},\quad N_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \tag{22}$$

Then by using Eqs. (12) and (22), the strains at the middle surface are expressed in terms of the stress function $\phi(x,y)$ and the deflection w:

$$\begin{Bmatrix}\varepsilon_{x_0}\\ \varepsilon_{y_0}\\ \gamma_{xy_0}\end{Bmatrix}=\begin{bmatrix}P_{11}&P_{12}&0\\P_{12}&P_{11}&0\\0&0&P_{66}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2\phi}{\partial y^2}\\ \dfrac{\partial^2\phi}{\partial x^2}\\ \dfrac{\partial^2\phi}{\partial x\partial y}\end{Bmatrix}+\begin{bmatrix}Q_{11}&Q_{12}&0\\Q_{12}&Q_{11}&0\\0&0&Q_{66}\end{bmatrix}\begin{Bmatrix}-\dfrac{\partial^2 w}{\partial x^2}\\ -\dfrac{\partial^2 w}{\partial y^2}\\ -2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix} \tag{23}$$

The bending moments are rearranged by substituting Eq. (23) into (13) as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} -Q_{11} & -Q_{12} & 0 \\ -Q_{12} & -Q_{11} & 0 \\ 0 & 0 & -Q_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 \phi}{\partial y^2} \\ \frac{\partial^2 \phi}{\partial x^2} \\ \frac{\partial^2 \phi}{\partial x \partial y} \end{Bmatrix} + \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{11} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{24}$$

where

$$\begin{aligned}
P_{11} &= A_{11}/\Delta \\
P_{12} &= -A_{12}/\Delta \\
P_{66} &= -1/A_{66} \\
Q_{11} &= (A_{12}B_{12} - A_{11}B_{11})/\Delta \\
Q_{12} &= (A_{12}B_{11} - A_{11}B_{12})/\Delta \\
Q_{66} &= -B_{66}/A_{66} \\
S_{11} &= B_{11}Q_{11} + B_{12}Q_{12} + C_{11} \\
S_{12} &= B_{11}Q_{12} + B_{12}Q_{11} + C_{12} \\
S_{66} &= C_{66} + B_{66}Q_{66} \\
\Delta &= A_{11}^2 - A_{12}^2
\end{aligned}$$

In order to find the relation between stress function $\phi(x,y)$ and deflection w, we substitute Eq. (23) into Eq. (21) and the equilibrium equation becomes:

$$Q_{12}\frac{\partial^4 \phi}{\partial x^4} + 2(Q_{11} - Q_{66})\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + Q_{12}\frac{\partial^4 \phi}{\partial y^4} + S_{11}\frac{\partial^4 w}{\partial x^4} + 2(S_{12} + 2S_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + S_{11}\frac{\partial^4 w}{\partial y^4} = q_z(x,y) \tag{25}$$

Since the stress function $\phi(x,y)$ and the deflection w in Eq. (25) are unknown, one more equation is needed. A compatibility equation is then used to provide the governing equation for $\phi(x,y)$. By using Eq. (7), the compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

can be expressed as

$$\frac{\partial^2 \varepsilon_{x_0}}{\partial y^2} + \frac{\partial^2 \varepsilon_{y_0}}{\partial x^2} = \frac{\partial^2 \gamma_{xy_0}}{\partial x \partial y} \tag{26}$$

Consequently, substituting Eq. (23) into Eq. (26) gives the relation between stress function $\phi(x,y)$ and the deflection w:

$$P_{11}\frac{\partial^4 \phi}{\partial x^4} + (2P_{12} - P_{66})\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + P_{11}\frac{\partial^4 \phi}{\partial y^4} - Q_{12}\frac{\partial^4 w}{\partial x^4} - 2(Q_{11} - Q_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} - Q_{12}\frac{\partial^4 w}{\partial y^4} = 0 \tag{27}$$

Eqs. (25) and (27) provide the simultaneous equations to solve the stress function $\phi(x,y)$ and the deflection w.

4. Solutions to simply supported rectangular FGM plates

Consider an FGM plate with length a, width b, and uniform thickness h subjected to the lateral load $q_z(x,y)$. Expanding the lateral load $q_z(x,y)$ by Fourier series:

$$q_z(x,y) = \sum\sum q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{28}$$

where

$$q_{mn} = \iint q_z(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, \mathrm{d}x\, \mathrm{d}y \tag{29}$$

If $q_z(x,y)$ is uniform distributed load, i.e., $q_z(x,y) = q_0$, then the quantity q_{mn} calculated by Eq. (29) is

$$q_{mn} = \begin{cases} \dfrac{16q_0}{\pi^2 mn} & m,n = 1,3,5,\ldots \\ 0 & m,n = 2,4,6,\ldots \end{cases} \tag{30}$$

If the load is a line load of $q_z(x,y) = P_0$ acting at $x = u$, then

$$q_{mn} = \begin{cases} \dfrac{8P_0}{a\pi n} \sin\dfrac{m\pi u}{a} & m,n = 1,3,5,\ldots \\ 0 & m,n = 2,4,6,\ldots \end{cases} \tag{31}$$

If the load is a point load of $q_z(x,y) = P$ acting at $x = u$, $y = v$, then

$$q_{mn} = \begin{cases} \dfrac{4P}{ab} \sin\dfrac{m\pi u}{a} \sin\dfrac{n\pi v}{b} & m,n = 1,3,5,\ldots \\ 0 & m,n = 2,4,6,\ldots \end{cases} \tag{32}$$

The boundary conditions of the simply supported rectangular FGM plate are

$$\begin{cases} w = 0 \\ M_x = 0 \end{cases} \quad \text{at } x = 0 \text{ and } x = a \tag{33a}$$

$$\begin{cases} w = 0 \\ M_y = 0 \end{cases} \quad \text{at } y = 0 \text{ and } y = b \tag{33b}$$

In order to satisfy the loading condition in Eq. (28) and the boundary conditions in Eqs. (33) the stress function $\phi(x,y)$ and the displacement w of the FGM plate should be of the form:

$$w(x,y) = \sum_m \sum_n w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{34a}$$

$$\phi(x,y) = \sum_m \sum_n \phi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{34b}$$

where w_{mn} and ϕ_{mn} are unknown constants and can be determined from equilibrium and compatibility equations. By substituting Eq. (34) into the equilibrium of Eq. (25) and the compatibility of Eq. (27) and solving the simultaneous equations, the coefficients of w_{mn} and ϕ_{mn} are obtained as

$$w_{mn} = \left(\frac{J}{JH + K^2}\right) q_{mn} \tag{35a}$$

$$\phi_{mn} = \left(\frac{K}{JH + K^2}\right) q_{mn} \tag{35b}$$

where

$$H = S_{11}\left(\frac{m\pi}{a}\right)^4 + 2(S_{12} + 2S_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + S_{11}\left(\frac{n\pi}{b}\right)^4 \tag{35c}$$

$$J = P_{11}\left(\frac{m\pi}{a}\right)^4 + (2P_{12} - P_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + P_{11}\left(\frac{n\pi}{b}\right)^4 \tag{35d}$$

$$K = Q_{12}\left(\frac{m\pi}{a}\right)^4 + 2(Q_{11} - Q_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + Q_{12}\left(\frac{n\pi}{b}\right)^4 \tag{35e}$$

With the aid of Eqs. (35) and (23), the strains at the middle surface are

$$\varepsilon_{x_0} = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left[(-P_{12}K + Q_{11}J)\left(\frac{m\pi}{a}\right)^2 + (-P_{11}K + Q_{12}J)\left(\frac{n\pi}{b}\right)^2\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{36a}$$

$$\varepsilon_{y_0} = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left[(-P_{12}K + Q_{11}J)\left(\frac{n\pi}{b}\right)^2 + (-P_{11}K + Q_{12}J)\left(\frac{m\pi}{a}\right)^2\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{36b}$$

$$\gamma_{xy_0} = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}(P_{66}K - 2Q_{66}J)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{36c}$$

Consequently, the strain and stress fields of the FGM plate can be obtained from Eqs. (7) and (8) and they are

$$\varepsilon_x = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left[(-P_{12}K + Q_{11}J + zJ)\left(\frac{m\pi}{a}\right)^2 + (-P_{11}K + Q_{12}J)\left(\frac{n\pi}{b}\right)^2\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{37a}$$

$$\varepsilon_y = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left[(-P_{12}K + Q_{11}J + zJ)\left(\frac{n\pi}{b}\right)^2 + (-P_{11}K + Q_{12}J)\left(\frac{m\pi}{a}\right)^2\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{37b}$$

$$\gamma_{xy} = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}(P_{66}K - 2Q_{66}J - 2zJ)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{37c}$$

and

$$\begin{aligned}
\sigma_x &= \frac{E(z)}{1 - v(z)^2}\sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left\{(-P_{12}K + Q_{11}J + zJ)\left[\left(\frac{m\pi}{a}\right)^2 + v(z)\left(\frac{n\pi}{b}\right)^2\right]\right.\\
&\quad \left. + (-P_{11}K + Q_{12}J)\left[\left(\frac{n\pi}{b}\right)^2 + v(z)\left(\frac{m\pi}{a}\right)^2\right]\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\sigma_y &= \frac{E(z)}{1 - v(z)^2}\sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left\{(-P_{12}K + Q_{11}J + zJ)\left[\left(\frac{n\pi}{b}\right)^2 + v(z)\left(\frac{m\pi}{a}\right)^2\right]\right.\\
&\quad \left. + (-P_{11}K + Q_{12}J)\left[\left(\frac{m\pi}{a}\right)^2 + v(z)\left(\frac{n\pi}{b}\right)^2\right]\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\tau_{xy} &= \frac{E(z)}{2[1 + v(z)]}\sum_m \sum_n \frac{q_{mn}}{JH + K^2}(P_{66}K - 2Q_{66}J - 2zJ)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)
\end{aligned} \tag{38abc}$$

The inplane axial forces and the bending moments of the FGM plate can be also obtained from Eqs. (22), (24), and (35) as

$$N_x = \frac{\partial^2\phi}{\partial y^2} = \sum_m \sum_n \frac{-Kq_{mn}}{JH + K^2}\left(\frac{n\pi}{b}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{39a}$$

$$N_y = \frac{\partial^2\phi}{\partial y^2} = \sum_m \sum_n \frac{-Kq_{mn}}{JH + K^2}\left(\frac{m\pi}{a}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{39b}$$

$$N_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \sum_m \sum_n \frac{-Kq_{mn}}{JH + K^2}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{39c}$$

and

$$M_x = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left[(Q_{12}K + S_{11}J)\left(\frac{m\pi}{a}\right)^2 + (Q_{11}K + S_{12}J)\left(\frac{n\pi}{b}\right)^2\right] \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{40a}$$

$$M_y = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}\left[(Q_{12}K + S_{11}J)\left(\frac{n\pi}{b}\right)^2 + (Q_{11}K + S_{12}J)\left(\frac{m\pi}{a}\right)^2\right] \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{40b}$$

$$M_{xy} = \sum_m \sum_n \frac{q_{mn}}{JH + K^2}(-KQ_{66} - 2JS_{66})\left(\frac{n\pi}{b}\right)\left(\frac{m\pi}{a}\right) \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{40c}$$

5. Solutions to the material with $E = E(z)$ only

If both the Young's modulus and the Poisson's ratio are considered for calculating the coefficient, the integration will turn out to be very complicate. Delale and Erdogan (1983) indicated that the influence of Poisson's ratio on the deformation of plates would be much less than that of the Young's modulus. Therefore, the solutions for the material with the Poisson's ratio being assumed to be constant and the Young's modulus varying in the thickness direction are derived in the following.

For the material with $v = \text{constant}$ and $E = E(z)$, it can be found that

$$\begin{aligned} &A_{12} = vA_{11}, \quad B_{12} = vB_{11}, \quad C_{12} = vC_{11} \\ &A_{66} = \left(\frac{1-v}{2}\right)A_{11}, \quad B_{66} = \left(\frac{1-v}{2}\right)B_{11}, \quad C_{66} = \left(\frac{1-v}{2}\right)C_{11} \\ &P_{12} = -vP_{11}, \quad P_{66} = -2(1+v)P_{11}, \quad Q_{12} = 0 \\ &Q_{66} = Q_{11}, \quad S_{12} = vS_{11}, \quad S_{66} = \frac{1-v}{2}S_{11} \end{aligned} \tag{41a–l}$$

Consequently, the parameters H, J, K, and $\widetilde{K}$ are

$$H = S_{11}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 \tag{42a}$$

$$J = P_{11}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 \tag{42b}$$

$$K = 0 \tag{42c}$$

with

$$P_{11} = \frac{1}{(1-v^2)A_{11}} \tag{42d}$$

$$Q_{11} = -\frac{B_{11}}{A_{11}} \tag{42e}$$

$$S_{11} = B_{11}Q_{11} + C_{11} \tag{42f}$$

Then the deflection of the FGM plates obtained by using of Eqs. (34), (35) and (42) is

$$w = \frac{1}{S_{11}} \sum_m \sum_n \frac{q_{mn}}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{43}$$

where S_{11} can be evaluated from Eqs. (42g) and (14). If the plate is homogeneous with isotropic material in which $E_1 = E_2 = E$, then the coefficient

$$S_{11} = \frac{Eh^3}{12(1-\nu^2)} \tag{44}$$

and the deflection can be simplified as

$$w(x,y) = \frac{12(1-\nu^2)}{Eh^3} \sum_m \sum_n \frac{q_{mn}}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{45}$$

Eq. (45) which is the solution of the homogeneous plates agrees with that in Timoshenko and Woinowsky-Krieger (1959). Noting that Eq. (43) is the Navier solution for a rectangular simply supported FGM plate.

Finally, the strains, stresses, axial forces and the bending moments for the FGM plates with the material of $E = E(z)$ only are

$$\varepsilon_{x_0} = \frac{Q_{11}}{S_{11}} \sum_m \sum_n \frac{q_{mn}\left(\frac{m\pi}{a}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{46a}$$

$$\varepsilon_{y_0} = \frac{Q_{11}}{S_{11}} \sum_m \sum_n \frac{q_{mn}\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{46b}$$

$$\gamma_{xy_0} = \frac{-2Q_{11}}{S_{11}} \sum_m \sum_n \frac{q_{mn}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{46c}$$

$$\varepsilon_x = \frac{Q_{11}+z}{S_{11}} \sum_m \sum_n \frac{q_{mn}\left(\frac{m\pi}{a}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{46d}$$

$$\varepsilon_y = \frac{Q_{11}+z}{S_{11}} \sum_m \sum_n \frac{q_{mn}\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{46e}$$

$$\gamma_{xy} = \frac{-2(Q_{11}+z)}{S_{11}} \sum_m \sum_n \frac{q_{mn}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{46f}$$

$$\sigma_x = \frac{E(z)}{(1-v^2)}\frac{(Q_{11}+z)}{S_{11}}\sum_m\sum_n q_{mn}\frac{\left(\frac{m\pi}{a}\right)^2+v\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{46g}$$

$$\sigma_y = \frac{E(z)}{(1-v^2)}\frac{(Q_{11}+z)}{S_{11}}\sum_m\sum_n q_{mn}\frac{\left(\frac{n\pi}{b}\right)^2+v\left(\frac{m\pi}{a}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{46h}$$

$$\tau_{xy} = \frac{-E(z)}{(1+v)}\frac{(Q_{11}+z)}{S_{11}}\sum_m\sum_n q_{mn}\frac{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{46i}$$

$$N_x = N_y = N_{xy} = 0 \tag{46j}$$

$$M_x = \sum_m\sum_n q_{mn}\frac{\left(\frac{m\pi}{a}\right)^2+v\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{46k}$$

$$M_y = \sum_m\sum_n q_{mn}\frac{\left(\frac{n\pi}{b}\right)^2+v\left(\frac{m\pi}{a}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{46l}$$

$$M_{xy} = \sum_m\sum_n q_{mn}\frac{(v-1)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{46m}$$

6. Series solutions to the P-, S-, and E-FGM plates

The series solutions of the FGM plates with materials of $v = \text{constant}$ and $E = E(z)$ have been obtained in Eqs. (43) and (46). However, the quantities in the series solutions involve integrations according to different material gradations. Hence, we will further derive the solution of FGM plates with specified volume reaction.

6.1. The series solution to P-FGM plates

volume fraction and the Young's modulus of the P-FGM plate are defined in Eqs. (1) and (2). By substituting Eq. (2) into Eq. (14), we obtain

$$A_{11} = \frac{h}{1-v^2}\left(\frac{pE_2+E_1}{p+1}\right) \tag{47a}$$

$$B_{11} = \frac{h^2}{1-v^2}\left[\frac{(E_1-E_2)p}{2(p+1)(p+2)}\right] \tag{47b}$$

$$C_{11} = \frac{h^3}{12(1-v^2)}\left[E_2-\frac{12(E_1-E_2)}{(P+2)(P+3)}\right] \tag{47c}$$

$$P_{11} = \frac{(p+1)}{h(pE_2 + E_1)} \tag{47d}$$

$$Q_{11} = \frac{-ph}{2(p+2)} \frac{(E_1 - E_2)}{(pE_2 + E_1)} \tag{47e}$$

$$S_{11} = \frac{h^3}{12(1-\nu^2)} \left[E_2 + \frac{3(p^2+p+2)(E_1 - E_2)}{(p+1)(p+2)(p+3)} - \frac{3p^2(E_1 - E_2)^2}{(p+1)(p+2)^2(pE_2 + E_1)} \right] \tag{47f}$$

Upon obtaining S_{11} and Q_{11} in Eqs. (47), the mechanical behaviors of the P-FGM plate can be investigated using Eq. (46) without any difficulty.

6.2. The solution to S-FGM plates

With the manner similar to the P-FGM plates, the coefficients A_{ij}, B_{ij} andC_{ij} of the S-FGM plates are expressed in terms of Young's moduli as follows:

$$A_{11} = \frac{h}{1-\nu^2} \left(\frac{E_1 + E_2}{2} \right) \tag{48a}$$

$$B_{11} = \frac{h^2(E_1 - E_2)}{8(1-\nu^2)} \frac{(p^2 + 3p)}{(p+1)(p+2)} \tag{48b}$$

$$C_{11} = \frac{h^3}{12(1-\nu^2)} \left[\frac{(E_1 + E_2)}{2} \right] \tag{48c}$$

and

$$P_{11} = \frac{2}{h(E_1 + E_2)} \tag{48d}$$

$$Q_{11} = \frac{-h(E_1 - E_2)(p^2 + 3p)}{4(E_1 + E_2)(p+1)(p+2)} \tag{48e}$$

$$S_{11} = \frac{h^3}{8(1-\nu^2)} \left[\frac{E_1 + E_2}{3} - \frac{(E_1 - E_2)^2(p^2 + 3p)^2}{4(E_1 + E_2)(p+1)^2(p+2)^2} \right] \tag{48f}$$

6.3. The solution to E-FGM plates

The parameters P_{11}, Q_{11}, and S_{11} of the E-FGM can be obtained as

$$P_{11} = \frac{1}{h} \cdot \frac{\ln(E_1/E_2)}{(E_1 - E_2)} \tag{49a}$$

$$Q_{11} = \frac{h}{\ln(E_1/E_2)} - \frac{h(E_1 + E_2)}{2(E_1 - E_2)} \tag{49b}$$

$$S_{11} = \frac{h^3}{1-\nu^2} \left\{ \frac{(E_1 - E_2)^2 - E_1 E_2 [\ln(E_1/E_2)]^2}{[\ln(E_1/E_2)]^3 (E_1 - E_2)} \right\} \tag{49c}$$

The closed-form solutions to P-, S-, and E-FGM plates have been derived. The displacements and stresses at any point inside the FGM plate can be easily calculated according to the given material properties and the geometric conditions.

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